

Asymptotic Profile of Solutions to the Linearized Compressible Navier-Stokes Flow

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Abstract

We consider the asymptotic behavior as $t \rightarrow +\infty$ of the L^2 -norm of the velocity of the linearized compressible Navier-Stokes equations in \mathbf{R}^n ($n \geq 2$). As an application we shall study the optimality of the decay rate for the L^2 -norm of the velocity by deriving the decay estimate from below as $t \rightarrow +\infty$.

1 Introduction

In this paper, we are concerned with the following linearized Compressible Navier-Stokes flow in \mathbf{R}^n with $n \geq 2$:

$$\rho_t(t, x) + \gamma \operatorname{div} v(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbf{R}^n, \quad (1.1)$$

$$v_t(t, x) - \alpha \Delta v(t, x) - \beta \nabla \operatorname{div} v(t, x) + \gamma \nabla \rho(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbf{R}^n, \quad (1.2)$$

$$\rho(0, x) = \rho_0(x), \quad v(0, x) = v_0(x), \quad x \in \mathbf{R}^n, \quad (1.3)$$

where α and β , the viscosity coefficient, are constants satisfying the thermodynamic restriction $\alpha > 0$ and $\beta \geq 0$. The constant coefficient γ is such that $\gamma > 0$ and

$$v(t, x) := {}^T(v_1(t, x), v_2(t, x), \dots, v_n(t, x))$$

is the vector valued unknown velocity of the fluid, $\rho(t, x)$ is a scalar valued unknown density of the fluid. Furthermore, $v_0(x) := {}^T(v_{01}(x), v_{02}(x), \dots, v_{0n}(x))$ and $\rho_0(x)$ are given initial data.

Concerning the L^p - L^q estimates of the solution $(\rho(t, x), v(t, x))$ to the linearized NS equation (1.1)-(1.3), one should first mention a precise result due to Kobayashi-Shibata [13], and in particular, they investigated the diffusion wave property of the solution in terms of the L^∞ -norm. The diffusion wave property was studied by Hoff-Zumbrum [5, 6] and Liu-Wang [15] to the (nonlinear) compressible Navier-Stokes flow. We should cite important closely related results due to Brezina-Kagei [1], Chowdhury-Ramaswamy [2], Decklnik [4] and the references therein as for the large time behavior of solutions to the compressible Navier-Stokes equations. In particular, we should mention the recent work by Ma-Wang [17], which studied a large time asymptotic behavior of contact wave for the Cauchy problem of the one-dimensional compressible

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Navier-Stokes equations with restriction to zero viscosity. Furthermore, Matsumura-Nishida [18] studied the existence theory and time-asymptotic L^2 decay of solutions to the original (nonlinear) Navier-Stokes systems, and this theory is generalized to the more general hyperbolic-parabolic systems by Kawashima [12]. On the other hand, it is well-known that the density $\rho(t, x)$ becomes a solution of a corresponding viscoelastic equation, and therefore, the decay estimates due to Shibata [20] and Ponce [19] are very useful to investigate the problem (1.1)-(1.3). All these investigations are quite restricted to the decay property of the solutions in terms of L^p -norms. But, to the best of authors' knowledge, it seems that there still does not exist a research from the viewpoint that catches an asymptotic profile itself of the solutions to problem (1.1)-(1.3). Recently, in Liu-Noh [14] they have announced new interesting results about the asymptotic profile and its point-wise decay estimates of the Green function to the (linearized) Navier-Stokes System. Furthermore, quite recently Ikehata-Onodera [10] has caught the explicit profile of the "density" $\rho(t, x)$ as $t \rightarrow +\infty$ through the study of the viscoelastic equations, which is inspired from the previous results by [9]. In this connection, the asymptotic profile of the viscoelastic equation was first discovered in the paper due to Ikehata-Todorova-Yordanov [11]. The profile is so called the diffusion wave, which is popular in the field of the Navier-Stokes systems. The result of [10] reads as follows.

Theorem 1.1 *Let $n \geq 2$. Then, it is true that there exist constants $C > 0$ and $\eta > 0$ such that for $t > 0$*

$$\begin{aligned} & \int_{\mathbf{R}^n} \left| \hat{\rho}(t, \xi) - \left[-(i\xi) \cdot P_0 e^{-\frac{(\alpha+\beta)t|\xi|^2}{2}} \frac{\sin(\gamma t|\xi|)}{|\xi|} + Q_0 e^{-\frac{(\alpha+\beta)t|\xi|^2}{2}} \cos(\gamma t|\xi|) \right] \right|^2 d\xi \\ & \leq C t^{-\frac{n}{2}-1} \|\rho_0\|_{1,1}^2 + C t^{-\frac{n}{2}-1} |P_0|^2 + C t^{-\frac{n}{2}-1} |Q_0|^2 + C t^{-\frac{n}{2}-1} \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) \\ & + C e^{-\eta t} (\|\operatorname{div} v_0\|^2 + \|\rho_0\|^2 + |Q_0|^2 + |P_0|^2), \end{aligned}$$

for $t \gg 1$, where

$$P_0 := (P_{01}, P_{02}, \dots, P_{0n}), \quad P_{0j} := \int_{\mathbf{R}^n} v_{0j}(x) dx \quad (j = 1, 2, \dots, n), \quad Q_0 := \int_{\mathbf{R}^n} \rho_0(x) dx$$

and $\hat{\rho}(t, \xi)$ is the usual Fourier transform of $\rho(t, x)$.

However, until now we still did not know about the explicit profile of the velocity $v(t, x)$ of the fluid.

The main purpose of this note is to announce the exact profile of the velocity $v(t, x)$ as $t \rightarrow +\infty$ (see Lemma 3.1 below).

Theorem 1.2 *Let $n \geq 2$. Then, it is true that there exists generous constants $C > 0$ and $\eta > 0$ such that*

$$\begin{aligned} & \int_{\mathbf{R}^n} \left| \hat{v}(t, \xi) - P_0 e^{-\alpha|\xi|^2 t} + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} \right. \\ & + \left. (i\xi) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \cos(\gamma t|\xi|) \right|^2 d\xi \\ & \leq C (|P_0|^2 + |Q_0|^2 + \sum_{j=1}^n \|v_{0j}\|_{1,1}^2 + \|\rho_0\|_{1,1}^2) t^{-\frac{n}{2}-1} + C e^{-\eta t} (\|v_0\|^2 + \|\rho_0\|^2), \end{aligned}$$

for large $t \gg 1$, where $C > 0$ depends only on γ, α, β , and $\hat{v}(t, \xi)$ is the Fourier transform of $v(t, x)$.

As an application, one has the optimal decay estimate of the L^2 -norm of the velocity $v(t, x)$ for the compressible fluid.

Theorem 1.3 *Let $n \geq 2$. Then, it is true that there exist constants $C_j > 0$ ($j = 1, 2$) such that for $t \gg 1$*

$$C_1 \left(|P_0| + |Q_0| \right) t^{-\frac{n}{4}} \leq \|v(t, \cdot)\| \leq C_2 \left(|P_0| + |Q_0| + \sum_{j=1}^n \|v_{0j}\|_{1,1} + \|\rho_0\|_{1,1} + \|v_0\| + \|\rho_0\| \right) t^{-\frac{n}{4}},$$

provided that $|Q_0| \neq 0$ and $|P_0|/|Q_0| \ll 1$.

Remark 1.1 It is still open to show the optimality above in the case when the assumptions $|Q_0| \neq 0$ and $|P_0|/|Q_0| \ll 1$ do not hold. However, the assumption $|P_0|/|Q_0| \ll 1$ means that $|P_0|/|Q_0| < \frac{1}{1+C_n}$ with $C_n > 0$ depending on the dimension n and the coefficients α, β , which can be calculated explicitly.

Notation. Throughout this paper, $\|\cdot\|_q$ stands for the usual $L^q(\mathbf{R}^n)$ -norm. For simplicity of notations, in particular, we use $\|\cdot\|$ instead of $\|\cdot\|_2$. Furthermore, we set

$$f \in L^{1,\gamma}(\mathbf{R}^n) \Leftrightarrow f \in L^1(\mathbf{R}^n), \|f\|_{1,\gamma} := \int_{\mathbf{R}^n} (1 + |x|^\gamma) |f(x)| dx < +\infty, \quad \gamma > 0,$$

$$\|f\| := \sqrt{\|f_1\|^2 + \cdots + \|f_n\|^2}$$

for $f = (f_1, \dots, f_n) \in (L^2(\mathbf{R}^n))^n$.

On the other hand, we denote the Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ by

$$\mathcal{F}(\phi)(\xi) := \mathcal{F}_{x \rightarrow \xi}(\phi)(\xi) := \hat{\phi}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-ix \cdot \xi} \phi(x) dx,$$

and we denote by $\mathcal{F}_{x \rightarrow \xi}^{-1}$ its usual inverse Fourier transform, where $i := \sqrt{-1}$, and $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$ for $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$. We also use the notation

$$v_t = \frac{\partial u}{\partial t}, \quad \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}, \quad \nabla f = \nabla f(x) := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right),$$

and the notation $\operatorname{div} v$ means the usual divergence of the vector valued function v .

Remark 1.2 One of our advantage is the ease of the method of our proof as compared with that of [13] when we catch the exact profile of the solution $(\rho(t, x), v(t, x))$. We use the method from [7] or [8] when one deals with the low frequency part of the Fourier transformed velocity, while in the high frequency region we shall rely on the method due to [16] combined with the Haraux-Komornik inequality (see Lemma 2.4 below), which is a version of the energy method in the Fourier space. The latter part seems much different from known techniques about the compressible Navier-Stokes equations.

Remark 1.3 In the case when (for example) $n = 3$, it follows from the study due to [14, p. 395] that

$$\begin{aligned} & \mathcal{F}_{x \rightarrow \xi}^{-1} \left(P_{0j} \frac{\xi_j \xi_k}{|\xi|^2} (\cos(\gamma t |\xi|) - 1) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \right) (x) \\ &= P_{0j} \frac{\partial^2}{\partial x_j \partial x_k} \frac{\gamma^2}{4\pi} \int_0^t \int_{|y|=1} G\left(\frac{\alpha+\beta}{2} t, x + \gamma \tau y\right) dS_y d\tau, \end{aligned}$$

and

$$\begin{aligned} & \mathcal{F}_{\mathbf{x} \rightarrow \xi}^{-1} \left(P_{0j} \frac{\xi_j \xi_k}{|\xi|^2} (e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} - e^{-\alpha|\xi|^2 t}) \right) (x) \\ &= P_{0j} \frac{\partial^2}{\partial x_j \partial x_k} \frac{\alpha - \beta}{2} \int_0^t \int_{|y|=1} G(\min\{\frac{\alpha + \beta}{2}, \alpha\}t + \frac{|\alpha - \beta|}{2}\tau, x + \gamma\tau y) dS_y d\tau, \end{aligned}$$

where $j, k = 1, 2, \dots, 3$, and

$$G(t, x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$$

is the 3-dimensional Gauss kernel. Therefore, if one can know the decomposition below:

$$\begin{aligned} & P_0 e^{-\alpha|\xi|^2 t} - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} - (i\xi) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \cos(\gamma t|\xi|) \\ &= P_0 e^{-\alpha|\xi|^2 t} - (i\xi) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \cos(\gamma t|\xi|) - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} \\ &= P_0 e^{-\alpha|\xi|^2 t} - (i\xi) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 \\ & \quad + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} \left(e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} - e^{-\alpha|\xi|^2 t} \right) + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} (\cos(\gamma t|\xi|) - 1) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \end{aligned} \tag{1.4}$$

one can get a precise profile of the velocity $v(t, x)$. This is left to the readers' check. Note that as for the one of profiles for (1.4) part in asymptotic sense, one can refer the reader to [8, p. 2167] in order to obtain the formula: for each $k = 1, 2, 3$ (cf. [13, Lemma 3.3])

$$\begin{aligned} & \mathcal{F}_{x \rightarrow \xi}^{-1} \left((i\xi_k) \hat{w}(t, \cdot) \hat{h}(t, \cdot) \right) (x) = \frac{\partial}{\partial x_k} (w(t, \cdot) * h(t, \cdot))(x) \\ &= C_0 (\alpha + \beta)^{-\frac{3}{2}} t^{-\frac{1}{2}} \frac{\partial}{\partial x_k} \int_{|z|=1} e^{-\frac{|x+tz|^2}{2(\alpha+\beta)t}} dS_z \end{aligned}$$

with some constant $C_0 > 0$. Here the function $w(t, x)$ is the fundamental solution to the free wave equation

$$\begin{aligned} & w_{tt}(t, x) - \gamma^2 \Delta w(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbf{R}^3, \\ & w(0, x) = 0, \quad w_t(0, x) = \delta(x), \quad x \in \mathbf{R}^3, \end{aligned}$$

where $\delta(x)$ is the usual Dirac measure, and

$$h(t, x) := (\alpha + \beta)^{-\frac{3}{2}} t^{-\frac{3}{2}} e^{-\frac{|x|^2}{2(\alpha+\beta)t}}.$$

It is easy to check

$$\hat{w}(t, \xi) = \frac{\sin(\gamma t|\xi|)}{|\xi|}, \quad \hat{h}(t, \xi) = e^{-\frac{(\alpha+\beta)t|\xi|^2}{2}}.$$

In the rest of this paper we shall give a proof of Theorem 1.2 in Section 2, and in Section 3 we will prove Theorem 1.3.

2 Proof of Theorem 1.2.

In this section, we shall prove Theorem 1.2 based on a device due to [10]. We first apply the Fourier transform to problem (1.1)–(1.3). Then the problem (1.1)–(1.3) can be reduced to the following system of ordinary differential equations with frequency parameter $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}_\xi^n$

$$\hat{\rho}_t(t, \xi) + i\gamma\xi \cdot \hat{v}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbf{R}_\xi^n, \quad (2.1)$$

$$\hat{v}_t(t, \xi) + \alpha|\xi|^2 \hat{v}(t, \xi) + \beta\xi(\xi \cdot \hat{v}(t, \xi)) + i\gamma\xi \hat{\rho}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbf{R}_\xi^n, \quad (2.2)$$

$$\hat{\rho}(0, \xi) = \hat{\rho}_0(\xi), \quad \hat{v}(0, \xi) = \hat{v}_0(\xi), \quad \xi \in \mathbf{R}_\xi^n. \quad (2.3)$$

We first prove the following lemma in the low frequency region in the case when $0 < |\xi| \ll 1$.

Lemma 2.1 *Let $n \geq 2$. Then, there exists a small $\delta_0 > 0$ such that*

$$\begin{aligned} & \int_{|\xi| \leq \frac{\delta_0}{\sqrt{2}}} \left| \hat{v}(t, \xi) - P_0 e^{-\alpha|\xi|^2 t} + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} \right. \\ & + \left. (i\xi) e^{-(\alpha+\beta)|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-(\alpha+\beta)|\xi|^2 t/2} \cos(\gamma t|\xi|) \right|^2 d\xi \\ & \leq C \left(|P_0|^2 + |Q_0|^2 + \sum_{j=1}^n \|v_{0j}\|_{1,1}^2 + \|\rho_0\|_{1,1}^2 \right) t^{-\frac{n}{2}-1}, \end{aligned}$$

for large $t \gg 1$, where $C > 0$ is a generous constant depending only on γ, α, β , and so on.

In order to prove Lemma 2.1 above, let us solve (2.1)–(2.3) directly from the viewpoint of the velocity $v(t, x)$ under the condition that $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}}$ with small $\delta_0 > 0$ by basing on the result due to [13, (2.9)]. In this case we get

$$\begin{aligned} \hat{v}(t, \xi) &= e^{-\alpha|\xi|^2 t} \hat{v}_0(\xi) - (i\gamma\xi) \left(\frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right) \hat{\rho}_0(\xi) \\ &+ \left(\frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} - e^{-\alpha|\xi|^2 t} \right) \frac{\xi(\xi \cdot \hat{v}_0(\xi))}{|\xi|^2}, \end{aligned} \quad (2.4)$$

provided that $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}}$, where $\sigma_j \in C$ ($j = 1, 2$) are given by the following expressions

$$\sigma_1 = \sigma_1(\xi) = \frac{-b|\xi|^2 + i|\xi|\sqrt{4a - b^2|\xi|^2}}{2}, \quad \sigma_2 = \sigma_2(\xi) = \frac{-b|\xi|^2 - i|\xi|\sqrt{4a - b^2|\xi|^2}}{2},$$

with

$$a := \gamma^2, \quad b := (\alpha + \beta), \quad \delta_0 := \frac{2\sqrt{a}}{b} = \frac{2\gamma}{\alpha + \beta}.$$

Now let us use an idea which was introduced in [7]. We consider the following decomposition of the initial data

$$\hat{v}_{0j}(\xi) = A_{0j}(\xi) - iB_{0j}(\xi) + P_{0j}, \quad (j = 1, 2, \dots, n), \quad (2.5)$$

$$\hat{\rho}_0(\xi) = A_\rho(\xi) - iB_\rho(\xi) + Q_0, \quad (2.6)$$

where

$$A_\rho(\xi) := \int_{\mathbf{R}^n} (\cos(x \cdot \xi) - 1) \rho_0(x) dx, \quad B_\rho(\xi) := \int_{\mathbf{R}^n} \sin(x \cdot \xi) \rho_0(x) dx,$$

$$A_{0j}(\xi) := \int_{\mathbf{R}^n} (\cos(x \cdot \xi) - 1) v_{0j}(x) dx, \quad B_{0j}(\xi) := \int_{\mathbf{R}^n} \sin(x \cdot \xi) v_{0j}(x) dx, \quad (j = 1, 2, \dots, n),$$

$$Q_0 := \int_{\mathbf{R}^n} \rho_0(x) dx, \quad P_{0j} := \int_{\mathbf{R}^n} v_{0j}(x) dx, \quad (j = 1, 2, \dots, n),$$

where v_{0j} are the components of the initial velocity v_0 .

Since we can write

$$(i\gamma)\xi \cdot \hat{v}_0(\xi) = (i\gamma) \sum_{j=1}^n \xi_j \cdot (A_{0j} - iB_{0j} + P_{0j}) =: (i\gamma\xi) \cdot (A_0(\xi) - iB_0(\xi) + P_0),$$

where

$$A_0(\xi) - iB_0(\xi) + P_0 := (A_{01}(\xi), A_{02}(\xi), \dots, A_{0n}(\xi)) - i(B_{01}(\xi), B_{02}(\xi), \dots, B_{0n}(\xi)) + (P_{01}, P_{02}, \dots, P_{0n}),$$

one has the following expression for the velocity $\hat{v}(t, \xi)$

$$\begin{aligned} \hat{v}(t, \xi) &= e^{-\alpha|\xi|^2 t} [A_0(\xi) - iB_0(\xi) + P_0] - (i\gamma\xi) \left(\frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right) [A_\rho(\xi) - iB_\rho(\xi) + Q_0] \\ &+ \left(\frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} - e^{-\alpha|\xi|^2 t} \right) \frac{\xi \left[\xi \cdot (A_0(\xi) - iB_0(\xi) + P_0) \right]}{|\xi|^2}, \end{aligned} \quad (2.7)$$

for all ξ satisfying $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}}$.

Now, it is easy to check that

$$\frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} = 2 \frac{e^{-bt|\xi|^2/2} \sin\left(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}\right)}{|\xi|\sqrt{4a-b^2|\xi|^2}}, \quad (2.8)$$

and

$$\frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} = -\frac{b|\xi| e^{-bt|\xi|^2/2} \sin\left(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}\right)}{\sqrt{4a-b^2|\xi|^2}} + e^{-bt|\xi|^2/2} \cos\left(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}\right). \quad (2.9)$$

So, it follows from (2.7), (2.8) and (2.9) that

$$\begin{aligned} \hat{v}(t, \xi) &= P_0 e^{-\alpha|\xi|^2 t} - \xi(\xi \cdot P_0) \frac{be^{-bt|\xi|^2/2} \sin\left(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}\right)}{|\xi|\sqrt{4a-b^2|\xi|^2}} \\ &+ \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-bt|\xi|^2/2} \cos\left(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}\right) \\ &- e^{-\alpha|\xi|^2 t} \frac{\xi(\xi \cdot P_0)}{|\xi|^2} - 2(i\gamma\xi) \frac{e^{-b|\xi|^2 t/2} \sin\left(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}\right)}{|\xi|\sqrt{4a-b^2|\xi|^2}} Q_0 + E_0(t, \xi), \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} E_0(t, \xi) &:= e^{-\alpha|\xi|^2 t} [A_0(\xi) - iB_0(\xi)] - (i\gamma\xi) \left(\frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right) [A_\rho(\xi) - iB_\rho(\xi)] \\ &+ \left(\frac{\sigma_1 e^{\sigma_2 t} - \sigma_2 e^{\sigma_1 t}}{\sigma_1 - \sigma_2} - e^{-\alpha|\xi|^2 t} \right) \frac{\xi \left(\xi \cdot [A_0(\xi) - iB_0(\xi)] \right)}{|\xi|^2}. \end{aligned} \quad (2.11)$$

Now, applying the mean value theorem for $|\xi| \leq \frac{\delta_0}{\sqrt{2}}$ it follows that

$$2 \frac{\sin(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2})}{|\xi|\sqrt{4a-b^2|\xi|^2}} = \frac{2}{\sqrt{4a-b^2|\xi|^2}} \frac{\sin(\sqrt{at}|\xi|)}{|\xi|} + t \left(\frac{\sqrt{4a-b^2|\xi|^2} - 2\sqrt{a}}{\sqrt{4a-b^2|\xi|^2}} \right) \cos(\varepsilon(t, \xi)) \quad (2.12)$$

and

$$\cos(\frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2}) = \cos(\sqrt{at}|\xi|) - t|\xi| \left(\frac{\sqrt{4a-b^2|\xi|^2} - 2\sqrt{a}}{2} \right) \sin(\eta(t, \xi)), \quad (2.13)$$

where

$$\begin{aligned} \varepsilon(t, \xi) &:= \frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2} \theta + \sqrt{at}|\xi|(1-\theta'), \\ \eta(t, \xi) &:= \frac{t|\xi|\sqrt{4a-b^2|\xi|^2}}{2} \theta' + \sqrt{at}|\xi|(1-\theta''), \end{aligned}$$

for some θ' and $\theta'' \in (0, 1)$. Furthermore, using again the mean value theorem it follows that

$$\frac{2}{\sqrt{4a-b^2|\xi|^2}} = \frac{1}{\sqrt{a}} + \frac{2b^2\theta|\xi|^2}{(4a-b^2\theta^2|\xi|^2)\sqrt{4a-b^2\theta^2|\xi|^2}}, \quad \theta \in (0, 1). \quad (2.14)$$

Then, from (2.10)–(2.14) in the case when $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}}$ we find the following useful expression for the Fourier transform of the velocity $v(t, x)$

$$\begin{aligned} \hat{v}(t, \xi) &= P_0 e^{-\alpha|\xi|^2 t} - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} - (i\xi) e^{-b|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 \\ &- \frac{b}{2} \xi(\xi \cdot P_0) e^{-b|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{\gamma|\xi|} + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-b|\xi|^2 t/2} \cos(\gamma t|\xi|) \\ &+ E_0(t, \xi) - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-b|\xi|^2 t/2} (t|\xi|) \frac{\sqrt{D} - 2\gamma}{2} \sin(\eta(t, \xi)) \\ &- (i\gamma\xi) Q_0 e^{-b|\xi|^2 t/2} \sin(\gamma t|\xi|) \frac{2b^2\theta|\xi|}{\sqrt{D_\theta^3}} - (i\gamma\xi) Q_0 t e^{-b|\xi|^2 t/2} \left(\frac{\sqrt{D} - 2\gamma}{\sqrt{D}} \right) \cos(\varepsilon(t, \xi)) \\ &- b^3 \xi(\xi \cdot P_0) \frac{\theta|\xi|}{\sqrt{D_\theta^3}} e^{-b|\xi|^2 t/2} \sin(\gamma t|\xi|) - b\xi(\xi \cdot P_0) \left(\frac{t}{2} \right) e^{-b|\xi|^2 t/2} \left(\frac{\sqrt{D} - 2\gamma}{\sqrt{D}} \right) \cos(\varepsilon(t, \xi)), \end{aligned} \quad (2.15)$$

where $D := 4a - b^2|\xi|^2$ and $D_\theta := 4a - b^2\theta^2|\xi|^2$.

In order to estimate the remainder term we set

$$\begin{aligned} E_1(t, \xi) &:= -\frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-b|\xi|^2 t/2} (t|\xi|) \frac{\sqrt{D} - 2\gamma}{2} \sin(\eta(t, \xi)), \\ E_2(t, \xi) &:= b^3 \xi(\xi \cdot P_0) \frac{\theta|\xi|}{\sqrt{D_\theta^3}} e^{-b|\xi|^2 t/2} \sin(\gamma t|\xi|), \\ E_3(t, \xi) &:= -b\xi(\xi \cdot P_0) \frac{t}{2} e^{-b|\xi|^2 t/2} \left(\frac{\sqrt{D} - 2\gamma}{\sqrt{D}} \right) \cos(\varepsilon(t, \xi)), \\ E_4(t, \xi) &:= -(i\gamma\xi) Q_0 e^{-b|\xi|^2 t/2} \sin(\gamma t|\xi|) \frac{2b^2\theta|\xi|}{\sqrt{D_\theta^3}}, \\ E_5(t, \xi) &:= -(i\gamma\xi) Q_0 t e^{-b|\xi|^2 t/2} \left(\frac{\sqrt{D} - 2\gamma}{\sqrt{D}} \right) \cos(\varepsilon(t, \xi)), \\ E_6(t, \xi) &:= -\frac{b}{2} \xi(\xi \cdot P_0) e^{-b|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{\gamma|\xi|}. \end{aligned}$$

In fact, for the profile that we consider, $\sum_{j=0}^6 E_j(t, \xi)$ is a remainder term. Moreover, the next calculations will show that it is the best choice. This implies that the Fourier transform of the velocity $v(t, x)$ is given by

$$\begin{aligned} \hat{v}(t, \xi) &= P_0 e^{-\alpha|\xi|^2 t} - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} - (i\xi) e^{-b|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 \\ &+ \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-b|\xi|^2 t/2} \cos(\gamma t|\xi|) + \sum_{j=0}^6 E_j(t, \xi) \end{aligned} \quad (2.16)$$

in the low frequency zone $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}}$.

Now, let us estimate all quantities $E_j(t, \xi)$ in terms of $L^2(\mathbf{R}_\xi^n)$ -norm in order to make sure the fact that $\sum_{j=0}^6 E_j(t, \xi)$ is the remainder term. For this ends, we first prepare the following elementary helpful estimate

$$|\sqrt{4a - b^2|\xi|^2} - 2\sqrt{a}| = \left| \frac{b^2|\xi|^2}{\sqrt{4a - b^2|\xi|^2} + 2\sqrt{a}} \right| \leq \frac{b^2|\xi|^2}{2\sqrt{a}}, \quad (2.17)$$

which hold for $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}} < \delta_0 = \frac{2\sqrt{a}}{b}$. We also note that

$$4a - b^2\theta^2|\xi|^2 \geq 4a - b^2|\xi|^2 \geq 2a$$

for $0 < |\xi| \leq \frac{\delta_0}{\sqrt{2}}$ and $0 < \theta < 1$.

Based on the inequalities (2.16) and (2.17), by using the Schwarz inequality we can proceed all estimates below except for $E_0(t, \xi)$.

$$\begin{aligned} \int_{|\xi| \leq \delta_0/\sqrt{2}} |E_1(t, \xi)|^2 d\xi &\leq t^2 |P_0|^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-b|\xi|^2 t} \frac{b^4|\xi|^4}{4a} d\xi \\ &\leq \frac{b^4}{4a} |P_0|^2 t^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^6 e^{-b|\xi|^2 t} d\xi \leq \frac{b^4}{4a} |P_0|^2 t^{2-\frac{n}{2}-1}, \end{aligned} \quad (2.18)$$

$$\begin{aligned} \int_{|\xi| \leq \delta_0/\sqrt{2}} |E_2(t, \xi)|^2 d\xi &\leq b^6 |P_0|^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^6 e^{-b|\xi|^2 t} \frac{1}{|4a - b^2\theta^2|\xi|^2|^3} d\xi \\ &\leq \frac{b^6 |P_0|^2}{(2a)^3} \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^6 e^{-b|\xi|^2 t} d\xi \leq \frac{b^6 |P_0|^2}{(2a)^3} t^{-\frac{n}{2}-3}, \end{aligned} \quad (2.19)$$

$$\begin{aligned} \int_{|\xi| \leq \delta_0/\sqrt{2}} |E_3(t, \xi)|^2 d\xi &\leq b^2 t^2 |P_0|^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^4 e^{-b|\xi|^2 t} \frac{|\sqrt{4a - b^2|\xi|^2} - 2\sqrt{a}|^2}{4a - b^2|\xi|^2} d\xi \\ &\leq b^2 |P_0|^2 t^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^4 e^{-b|\xi|^2 t} \frac{b^4|\xi|^4}{4a(4a - b^2|\xi|^2)} d\xi \\ &\leq \frac{b^6 |P_0|^2 t^2}{8a^2} \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^8 e^{-b|\xi|^2 t} d\xi \leq \frac{b^6 |P_0|^2 t^2}{8a^2} t^{-\frac{n}{2}-2}, \end{aligned} \quad (2.20)$$

$$\begin{aligned}
\int_{|\xi| \leq \delta_0/\sqrt{2}} |E_4(t, \xi)|^2 d\xi &\leq 4\gamma^2 b^4 |Q_0|^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^4 \frac{e^{-b|\xi|^2 t}}{(4a - b^2 \theta^2 |\xi|^2)^3} d\xi \\
&\leq \frac{4\gamma^2 b^4 |Q_0|^2}{|3a|^3} \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^4 e^{-b|\xi|^2 t} d\xi \\
&\leq \frac{4\gamma^2 b^4 |Q_0|^2}{|2a|^3} t^{-\frac{n}{2}-2},
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\int_{|\xi| \leq \delta_0/\sqrt{2}} |E_5(t, \xi)|^2 d\xi &\leq \gamma^2 t^2 |Q_0|^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-b|\xi|^2 t} \frac{|\sqrt{4a - b^2 |\xi|^2} - 2\sqrt{a}|^2}{4a - b^2 |\xi|^2} d\xi \\
&\leq \frac{\gamma^2 t^2 |Q_0|^2}{2a} \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-b|\xi|^2 t} \frac{b^4 |\xi|^4}{4a} d\xi \\
&\leq \frac{b^4 \gamma^2 |Q_0|^2}{8a^2} t^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^6 e^{-b|\xi|^2 t} d\xi \leq \frac{b^4 \gamma^2 |Q_0|^2}{8a^2} t^{-\frac{n}{2}-1},
\end{aligned} \tag{2.22}$$

$$\int_{|\xi| \leq \delta_0/\sqrt{2}} |E_6(t, \xi)|^2 d\xi \leq \frac{b^2}{4\gamma^2} |P_0|^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-b|\xi|^2 t} d\xi \leq \frac{b^2}{4\gamma^2} |P_0|^2 t^{-\frac{n}{2}-1}. \tag{2.23}$$

In the above estimates we also have used the elementary estimate

$$\int_{|\xi| \leq \frac{\delta_0}{\sqrt{2}}} |\xi|^k e^{-\lambda |\xi|^2 t} d\xi \leq C_n t^{-\frac{n+k}{2}}, \quad t > 0, \tag{2.24}$$

for $\lambda > 0$, $k + n > 0$ where C_n is a positive constant depending on δ_0 , λ , k and the dimension n .

In order to estimate $E_0(t, \xi)$, we prepare the following simple lemma which plays an essential role in this note. This idea has its origin in [7, Lemma 3.1].

Lemma 2.2 *Let $n \geq 1$. Then it holds that*

$$|A_\rho(\xi)| \leq L|\xi| \|\rho_0\|_{1,1}, \quad |A_{0j}(\xi)| \leq L|\xi| \|v_{0j}\|_{1,1}, \quad (j = 1, 2, \dots, n),$$

$$|B_\rho(\xi)| \leq M|\xi| \|\rho_0\|_{1,1}, \quad |B_{0j}(\xi)| \leq M|\xi| \|v_{0j}\|_{1,1}, \quad (j = 1, 2, \dots, n),$$

for all $\xi \in \mathbf{R}^n$, where

$$L := \sup_{\theta \neq 0} \frac{|1 - \cos \theta|}{|\theta|} < +\infty, \quad M := \sup_{\theta \neq 0} \frac{|\sin \theta|}{|\theta|} < +\infty.$$

We note that $L < 1$ and $M = 1$.

The estimate for $E_0(t, \xi)$ based on Lemma 2.2 is crucial in this paper.

In fact, it follows from (2.8), (2.9), (2.11), Lemma 2.2 and the Schwarz inequality that

$$\begin{aligned}
\int_{|\xi| \leq \delta_0/\sqrt{2}} |E_0(t, \xi)|^2 d\xi &\leq C \int_{|\xi| \leq \delta_0/\sqrt{2}} e^{-2a|\xi|^2 t} \left[|A_0(\xi)|^2 + |B_0(\xi)|^2 \right] d\xi \\
&\quad + C\gamma^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} e^{-b|\xi|^2 t} |\xi|^2 \left[|A_\rho(\xi)|^2 + |B_\rho(\xi)|^2 \right] \frac{1}{|\xi|^2 (4a - b^2 |\xi|^2)} d\xi
\end{aligned}$$

$$\begin{aligned}
& + C \int_{|\xi| \leq \delta_0/\sqrt{2}} \left(\frac{b^2 |\xi|^2 e^{-b|\xi|^2 t}}{4a - b^2 |\xi|^2} + e^{-b|\xi|^2 t} + e^{-2\alpha |\xi|^2 t} \right) \frac{|\xi|^4 [|A_0(\xi)|^2 + |B_0(\xi)|^2]}{|\xi|^4} d\xi \\
& \leq C(L^2 + M^2) \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-2\alpha |\xi|^2 t} d\xi \\
& + \frac{C\gamma^2(L^2 + M^2)}{2a} \|\rho_0\|_{1,1}^2 \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-b|\xi|^2 t} d\xi \\
& + \frac{Cb^2}{2a} \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-b|\xi|^2 t} [|A_0(\xi)|^2 + |B_0(\xi)|^2] d\xi \\
& + C \int_{|\xi| \leq \delta_0/\sqrt{2}} e^{-\min\{b, 2\alpha\} |\xi|^2 t} [|A_0(\xi)|^2 + |B_0(\xi)|^2] d\xi,
\end{aligned}$$

because of the fact that $4a - b^2 |\xi|^2 \geq 2a$ for $|\xi| \leq \frac{\delta_0}{\sqrt{2}}$.

The previous inequality and (2.24) imply that

$$\begin{aligned}
\int_{|\xi| \leq \delta_0/\sqrt{2}} |E_0(t, \xi)|^2 d\xi & \leq C(L^2 + M^2) \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) t^{-\frac{n}{2}-1} + \frac{C\gamma^2(L^2 + M^2)}{2a} \|\rho_0\|_{1,1}^2 t^{-\frac{n}{2}-1} \\
& + \frac{Cb^2(L^2 + M^2)}{2a} \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^4 e^{-b|\xi|^2 t} d\xi \\
& + C(L^2 + M^2) \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) \int_{|\xi| \leq \delta_0/\sqrt{2}} |\xi|^2 e^{-\min\{b, 2\alpha\} |\xi|^2 t} d\xi \tag{2.25} \\
& \leq C(L^2 + M^2) \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) t^{-\frac{n}{2}-1} + \frac{C\gamma^2(L^2 + M^2)}{2a} \|\rho_0\|_{1,1}^2 t^{-\frac{n}{2}-1} \\
& + \frac{Cb^2(L^2 + M^2)}{2a} \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) t^{-\frac{n}{2}-2} + C(L^2 + M^2) \left(\sum_{j=1}^n \|v_{0j}\|_{1,1}^2 \right) t^{-\frac{n}{2}-1},
\end{aligned}$$

where the constants after the last inequality depend on the dimension n , and they can be calculated explicitly.

The expression for $\hat{v}(t, \xi)$ in (2.16) combined with the estimates (2.18)–(2.23) and (2.25) above and the fact that $b = \alpha + \beta$, imply the statement of Lemma 2.1 in the low frequency region. \square

Finally in this section, we shall derive decay estimates in the high frequency region by relying on a special multiplier method in the Fourier space introduced in Charão-daLuz-Ikehata [3] (see also [16]).

Lemma 2.3 *Let $n \geq 2$. Then, there exists a constant $\eta > 0$ such that*

$$\int_{|\xi| \geq \frac{\delta_0}{\sqrt{2}}} |\hat{v}(t, \xi)|^2 d\xi \leq C(\|v_0\|^2 + \|\rho_0\|^2) e^{-\eta t} \quad (t \gg 1),$$

where $\delta_0 := \frac{2\gamma}{\alpha + \beta} = \frac{2\sqrt{a}}{b} > 0$.

Proof. We prove the lemma for any arbitrary high frequency zone, that is, for any fixed number $\lambda_0 > 0$ in place of $\frac{\delta_0}{\sqrt{2}}$ by using the multiplier method combined with a simple version of the Komornik lemma. This result is stronger than that of Kobayashi–Shibata [13], which proves similar results in high frequency zone only with $\lambda_0 \gg 1$.

In the proof of this lemma, in order to simplify the notation, we use $\hat{\rho}$ and \hat{v} in place of $\hat{\rho}(t, \xi)$ and $\hat{v}(t, \xi)$ and the same for \hat{v}_t and $\hat{\rho}_t$, respectively. Moreover, we can use $\hat{v}(S)$ and $\hat{\rho}(S)$ in place of $\hat{v}(S, \xi)$ and $\hat{\rho}(S, \xi)$, respectively.

Multiply equation (2.1) by $\hat{\rho}$ and equation (2.2) by \hat{v} . Then we obtain

$$\frac{d}{dt} \left(\frac{|\hat{\rho}|^2 + |\hat{v}|^2}{2} \right) + \alpha |\xi|^2 |\hat{v}|^2 + \beta |\xi \cdot \hat{v}|^2 + 2i \operatorname{Re}(\gamma \bar{\hat{\rho}} \hat{v} \cdot \xi) = 0.$$

The above identity says that $\operatorname{Re}(\gamma \bar{\hat{\rho}} \hat{v} \cdot \xi) = 0$. Thus, integrating the identity above on $[S, T]$ we get

$$\left(\frac{|\hat{\rho}|^2 + |\hat{v}|^2}{2} \right)_S^T + \alpha \int_S^T |\xi|^2 |\hat{v}|^2 dt + \beta \int_S^T |\xi \cdot \hat{v}|^2 dt = 0, \quad (2.26)$$

for all $0 < S < T$ and $\xi \in \mathbf{R}^n$. Then it follows that

$$\alpha \int_S^T |\xi|^2 |\hat{v}|^2 dt \leq \frac{|\hat{\rho}(S)|^2 + |\hat{v}(S)|^2}{2}, \quad 0 < S < T, \quad \xi \in \mathbf{R}^n. \quad (2.27)$$

In particular,

$$E(T) + \alpha \int_S^T |\xi|^2 |\hat{v}|^2 dt + \beta \int_S^T |\xi \cdot \hat{v}|^2 dt = E(S), \quad 0 < S < T, \quad \xi \in \mathbf{R}^n,$$

where

$$E(t) = E(t, \xi) = \frac{|\hat{\rho}(t, \xi)|^2 + |\hat{v}(t, \xi)|^2}{2}.$$

Multiplying the equation (2.2) by $\xi \bar{\hat{\rho}}$ we obtain

$$\xi \cdot \left(\hat{v}_t \bar{\hat{\rho}} + \alpha |\xi|^2 \hat{v} \bar{\hat{\rho}} + \beta \xi (\xi \cdot \hat{v}) \bar{\hat{\rho}} + i \gamma \xi |\hat{\rho}|^2 \right) = 0$$

or

$$(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} + \alpha |\xi|^2 (\xi \cdot \hat{v}) \bar{\hat{\rho}} + \beta |\xi|^2 (\xi \cdot \hat{v}) \bar{\hat{\rho}} + i \gamma |\xi|^2 |\hat{\rho}|^2 = 0.$$

Now, multiplying by $i = \sqrt{-1}$ it results

$$i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} + i \alpha |\xi|^2 (\xi \cdot \hat{v}) \bar{\hat{\rho}} + i \beta |\xi|^2 (\xi \cdot \hat{v}) \bar{\hat{\rho}} = \gamma |\xi|^2 |\hat{\rho}|^2.$$

Then, by integrating it over $[S, T]$ one gets

$$\gamma \int_S^T |\xi|^2 |\hat{\rho}|^2 dt = \int_S^T \left[i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} + i \alpha |\xi|^2 (\xi \cdot \hat{v}) \bar{\hat{\rho}} + i \beta |\xi|^2 (\xi \cdot \hat{v}) \bar{\hat{\rho}} \right] dt, \quad (2.28)$$

for $0 < S < T$ and $\xi \in \mathbf{R}^n$. Since $\gamma > 0$, the equation (2.1) says that $i(\xi \cdot \hat{v}) = -\frac{1}{\gamma} \hat{\rho}_t$. So, by substituting this fact in identity (2.28) it follows that

$$\begin{aligned} \gamma \int_S^T |\xi|^2 |\hat{\rho}|^2 dt &= \int_S^T \left[i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} - \alpha |\xi|^2 \frac{1}{\gamma} \hat{\rho}_t \bar{\hat{\rho}} - \beta |\xi|^2 \frac{1}{\gamma} \hat{\rho}_t \bar{\hat{\rho}} \right] dt \\ &= \int_S^T \left[i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} - \left(\frac{\alpha}{\gamma} + \frac{\beta}{\gamma} \right) |\xi|^2 \frac{d}{dt} (|\hat{\rho}|^2) \right] dt \\ &= \int_S^T i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} dt - \left[\frac{\alpha + \beta}{\gamma} |\xi|^2 |\hat{\rho}(t)|^2 \right]_S^T. \end{aligned}$$

Thus, one has

$$\gamma \int_S^T |\xi|^2 |\hat{\rho}|^2 dt \leq \int_S^T i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} dt + \frac{\alpha + \beta}{\gamma} |\xi|^2 |\hat{\rho}(S)|^2, \quad (2.29)$$

for $0 < S < T$ and $\xi \in \mathbf{R}^n$.

In order to estimate the integral in the right hand side of (2.29) we can use two possibilities: integration by parts or taking the conjugate of the equation (2.2) and multiplying by \hat{v}_t . If we employ the second option we get

$$|\hat{v}_t|^2 + \alpha |\xi|^2 \bar{\hat{v}} \cdot \hat{v}_t + \beta (\xi \cdot \hat{v}_t) (\xi \cdot \bar{\hat{v}}) + i\gamma \xi \bar{\hat{\rho}} \cdot \hat{v}_t = 0$$

or

$$|\hat{v}_t|^2 + \alpha |\xi|^2 \frac{d}{dt} \frac{|\hat{v}|^2}{2} + \beta \frac{d}{dt} \frac{|\xi \cdot \hat{v}|^2}{2} + i\gamma (\xi \cdot \hat{v}_t) \bar{\hat{\rho}} = 0.$$

By integrating the above identity on $[S, T]$ we have

$$\int_S^T |\hat{v}_t|^2 dt + \left[\alpha |\xi|^2 \frac{|\hat{v}|^2}{2} + \beta \frac{|\xi \cdot \hat{v}|^2}{2} \right]_S^T + \int_S^T i\gamma (\xi \cdot \hat{v}_t) \bar{\hat{\rho}} dt = 0,$$

which implies

$$\int_S^T i\gamma (\xi \cdot \hat{v}_t) \bar{\hat{\rho}} dt \leq \left[\alpha |\xi|^2 \frac{|\hat{v}|^2}{2} + \beta \frac{|\xi \cdot \hat{v}|^2}{2} \right]_{t=S} \leq C_{\alpha, \beta} |\xi|^2 |\hat{v}(S)|^2,$$

where $C_{\alpha, \beta} > 0$ is a constant depending only on α or β . Then it follows that

$$\int_S^T i(\xi \cdot \hat{v}_t) \bar{\hat{\rho}} dt \leq \frac{C_{\alpha, \beta}}{\gamma} |\xi|^2 |\hat{v}(S)|^2, \quad 0 < S < T, \quad \xi \in \mathbf{R}^n. \quad (2.30)$$

By substituting (2.30) into (2.29) one has obtained

$$\gamma \int_S^T |\xi|^2 |\hat{\rho}|^2 dt \leq \frac{C_{\alpha, \beta}}{\gamma} |\xi|^2 |\hat{v}(S)|^2 + \frac{\alpha + \beta}{\gamma} |\xi|^2 |\hat{\rho}(S)|^2, \quad (2.31)$$

for $\xi \in \mathbf{R}^n$ and $0 < S < T$. By combining (2.31) and (2.27) we arrive at

$$\begin{aligned} \int_S^T |\xi|^2 [|\hat{v}|^2 + |\hat{\rho}|^2] dt &\leq \frac{C_{\alpha, \beta}}{\gamma^2} |\xi|^2 |\hat{v}(S)|^2 + \frac{\alpha + \beta}{\gamma^2} |\xi|^2 |\hat{\rho}(S)|^2 \\ &\quad + \frac{|\xi|^2 |\hat{\rho}(S)|}{\lambda_0^2 2\alpha} + \frac{|\xi|^2 |\hat{v}(S)|}{\lambda_0^2 2\alpha}, \end{aligned}$$

for all $|\xi| \geq \lambda_0 > 0$ and $0 < S < T$. Then, we have obtained the following important estimate

$$\int_S^T [|\hat{v}|^2 + |\hat{\rho}|^2] dt \leq C_{\alpha, \beta, \gamma, \lambda_0} [|\hat{v}(S)|^2 + |\hat{\rho}(S)|^2], \quad (2.32)$$

for all $|\xi| \geq \lambda_0 > 0$ and $0 < S < T$, where $C_{\alpha, \beta, \gamma, \lambda_0} > 0$ is a positive constant depending on α , β , γ and δ_0 .

By using the definition of the energy for the system (2.1)–(2.2) in the Fourier space

$$E(t, \xi) := |\hat{v}(t, \xi)|^2 + |\hat{\rho}(t, \xi)|^2,$$

(2.32) implies that

$$\int_S^T \int_{|\xi| \leq \lambda_0} E(t, \xi) d\xi dt \leq C_{\alpha, \beta, \gamma, \lambda_0} \int_{|\xi| \leq \lambda_0} E(S, \xi) d\xi, \quad (2.33)$$

for $0 < S < T < \infty$.

Now, if we define the energy in high frequency zone in the Fourier space by

$$E_h(t) := \int_{|\xi| \leq \lambda_0} E(t, \xi) d\xi,$$

the estimate (2.33) says that

$$\int_S^\infty E_h(t) dt \leq C_{\alpha, \beta, \gamma, \lambda_0} E_h(S), \quad (2.34)$$

for all $S \geq 0$.

□

To get the final estimate for the energy just defined above on the high frequency zone, we use a simple version of the following well-known Haraux–Komornik lemma.

Lemma 2.4 *Let $E : [0, +\infty) \rightarrow [0, +\infty)$ be a non-increasing function and assume that there exists a constant $T_0 > 0$ such that*

$$\int_S^\infty E(t) dt \leq T_0 E(S),$$

for all $S \geq 0$. Then, it is true that

$$E(t) \leq E(0) e^{1 - \frac{t}{T_0}}$$

for all $t \geq T_0$.

Proof of Lemma 2.3 completed. In order to finalize the proof of Lemma 2.3, we note the energy in the high frequency region $E_h(t)$ is a non-increasing function of t due to the identity (2.26). Then, we can combine the estimate (2.34) and Lemma 2.4 to conclude that

$$E_h(t) \leq C E_h(0) e^{-\eta t} \leq C \left(\|v_0\|^2 + \|\rho_0\|^2 \right) e^{-\eta t}$$

for $\eta = \frac{1}{C_{\alpha, \beta, \gamma, \lambda_0}}$ and $t \geq T_0 := C_{\alpha, \beta, \gamma, \lambda_0}$, where C is a positive constant depending only on the coefficients of the system (1.1)–(1.2) and λ_0 . In particular, the above inequality proves the desired lemma with $\lambda_0 = \frac{\delta_0}{\sqrt{2}}$. □

Proof of Theorem 1.2. The proof of Theorem 1.2 is a direct consequence of Lemmas 2.1 and 2.3. □

3 Proof of Theorem 1.3.

In this section, we shall give a proof of Theorem 1.3. For this ends it suffices to get the following lemma because of the Plancherel theorem together with Theorem 1.2 and the useful inequality:

$$\begin{aligned} \|v(t, \cdot)\| &= \|\hat{v}(t, \cdot)\| \geq \left\| P_0 e^{-\alpha|\xi|^2 t} - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} \right. \\ &\quad - (i\xi) e^{-(\alpha+\beta)|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-(\alpha+\beta)|\xi|^2 t/2} \cos(\gamma t|\xi|) \left. \right\| \\ &= \left\| \hat{v}(t, \xi) - P_0 e^{-\alpha|\xi|^2 t} + \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} \right. \end{aligned}$$

$$\begin{aligned}
& + \left\| (i\xi)e^{-(\alpha+\beta)|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-(\alpha+\beta)|\xi|^2 t/2} \cos(\gamma t|\xi|) \right\| \\
& = \left\| (i\xi) \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 e^{-(\alpha+\beta)|\xi|^2 t/2} - \left\{ P_0 - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} \right\} e^{-\alpha|\xi|^2 t} \right. \\
& \quad \left. - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} \cos(\gamma t|\xi|) e^{-(\alpha+\beta)|\xi|^2 t/2} \right\| + O(t^{-\frac{n}{4}-\frac{1}{2}}) \\
& = \|X(t, \cdot) + Y(t, \cdot)\| + O(t^{-\frac{n}{4}-\frac{1}{2}}) \\
& \geq \|X(t, \cdot)\| - \|Y(t, \cdot)\| + O(t^{-\frac{n}{4}-\frac{1}{2}}),
\end{aligned} \tag{3.1}$$

where we have just defined as

$$\begin{aligned}
X(t, \xi) &:= (i\xi) \frac{\sin(\gamma t|\xi|)}{|\xi|} Q_0 e^{-(\alpha+\beta)|\xi|^2 t/2}, \\
Y(t, \xi) &:= -\left\{ P_0 - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} \right\} e^{-\alpha|\xi|^2 t} - \frac{\xi(\xi \cdot P_0)}{|\xi|^2} \cos(\gamma t|\xi|) e^{-(\alpha+\beta)|\xi|^2 t/2}.
\end{aligned}$$

Now we will prove the following lemma.

Lemma 3.1 *Let $n \geq 1$. Then, there exist three real numbers $C_j > 0$ ($j = 1, 2, 3$) depending on n , α or $\alpha + \beta$, such that*

$$\begin{aligned}
(1) \quad & C_1^2 |P_0|^2 t^{-\frac{n}{2}} \leq \int_{\mathbf{R}_\xi^n} \left| \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\alpha|\xi|^2 t} \right|^2 d\xi \leq C_1^{-2} |P_0|^2 t^{-\frac{n}{2}}, \\
(2) \quad & C_2^2 t^{-\frac{n}{2}} \leq \int_{\mathbf{R}_\xi^n} \left| (i\xi) e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \frac{\sin(\gamma t|\xi|)}{|\xi|} \right|^2 d\xi \leq C_2^{-2} t^{-\frac{n}{2}}, \\
(3) \quad & C_3^2 |P_0|^2 t^{-\frac{n}{2}} \leq \int_{\mathbf{R}_\xi^n} \left| \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-\frac{(\alpha+\beta)|\xi|^2 t}{2}} \cos(\gamma t|\xi|) \right|^2 d\xi \leq C_3^{-2} |P_0|^2 t^{-\frac{n}{2}},
\end{aligned}$$

for large $t \gg 1$.

Remark 3.1 It is well known that $C_4 t^{-\frac{n}{4}} \leq \|e^{-\alpha|\xi|^2 t}\| \leq C_4^{-1} t^{-\frac{n}{4}}$ as $t \rightarrow +\infty$ with some constant $C_4 > 0$ depending on $\alpha > 0$ and the dimension n .

Let us postpone the proof of Lemma 3.1 for a while. Once Lemma 3.1 could be proved, one can proceed the proof of Theorem 1.3 as follows.

Proof of Theorem 1.3. First, from Lemma 3.1 one can get

$$\|X(t, \cdot)\| \geq C_2 |Q_0| t^{-\frac{n}{4}}, \tag{3.2}$$

$$\|Y(t, \cdot)\| \leq C_* |P_0| t^{-\frac{n}{4}}, \tag{3.3}$$

with some constant $C_* > 0$ which depends on $C_j > 0$ ($j = 1, 3, 4$). Thus, it follows from (3.1), (3.2) and (3.3) that

$$\|v(t, \cdot)\| \geq (C_2 |Q_0| - C_* |P_0|) t^{-\frac{n}{4}} + o(t^{-\frac{n}{4}}).$$

So, if $|Q_0| \neq 0$, and $\frac{C_2}{2} |Q_0| > (\frac{C_2}{2} + C_*) |P_0|$, then as $t \rightarrow +\infty$ one can get

$$\|v(t, \cdot)\| \geq \frac{1}{2} (C_2 |Q_0| - C_* |P_0|) t^{-\frac{n}{4}} \geq \frac{C_2}{4} (|Q_0| + |P_0|) t^{-\frac{n}{4}},$$

which implies the desired estimate from below. The estimate from above is a direct consequence of Lemma 3.1, Remark 3.1, and Theorem 1.2. \square

Proof of Lemma 3.1. The estimate from above of item (1) easily follows from Remark 3.1 (see also (2.24)). The estimate from above of item (3) is estimated as follows because of the Schwarz inequality,

$$\begin{aligned} & \int_{\mathbf{R}_\xi^n} \left| \frac{\xi(\xi \cdot P_0)}{|\xi|^2} e^{-(\alpha+\beta)|\xi|^2 t/2} \cos(\gamma t|\xi|) \right|^2 d\xi \\ & \leq |P_0|^2 \int_{\mathbf{R}_\xi^n} e^{-(\alpha+\beta)|\xi|^2 t} d\xi \leq C_3^{-2} |P_0|^2 t^{-\frac{n}{2}}. \end{aligned}$$

Concerning the estimate from above of (2) is also an easy exercise, so we omit its check (see Remark 3.1).

About the estimate from below of (2), one follows a device from [8]. For this check, we set

$$I(t) := \int_{\mathbf{R}_\xi^n} |(i\xi) e^{-(\alpha+\beta)|\xi|^2 t/2} \frac{\sin(\gamma t|\xi|)}{|\xi|}|^2 d\xi.$$

Then, easily one can get a series of equalities below with the help of polar co-ordinate transform

$$\begin{aligned} I(t) &= \left(\int_{|\omega|=1} d\omega \right) (\alpha + \beta)^{-\frac{n}{2}} t^{-\frac{n}{2}} \int_0^\infty e^{-\theta^2} \theta^{n-1} \sin^2\left(\frac{\gamma}{\sqrt{\alpha+\beta}} \sqrt{t}\theta\right) d\theta \\ &= \frac{S_0}{2} \left(\int_{|\omega|=1} d\omega \right) (\alpha + \beta)^{-\frac{n}{2}} t^{-\frac{n}{2}} \\ &\quad - \frac{1}{2} \left(\int_{|\omega|=1} d\omega \right) (\alpha + \beta)^{-\frac{n}{2}} t^{-\frac{n}{2}} \int_0^\infty e^{-\theta^2} \theta^{n-1} \cos\left(\frac{2\gamma}{\sqrt{\alpha+\beta}} \sqrt{t}\theta\right) d\theta, \end{aligned} \quad (3.4)$$

where

$$S_0 = \int_0^\infty e^{-\theta^2} \theta^{n-1} d\theta.$$

Since

$$\lim_{t \rightarrow +\infty} \int_0^\infty e^{-\theta^2} \theta^{n-1} \cos\left(\frac{2\gamma}{\sqrt{\alpha+\beta}} \sqrt{t}\theta\right) d\theta = 0,$$

because of the fact $e^{-\theta^2} \theta^{n-1} \in L^1(0, \infty)$ and the Riemann-Lebesgue Theorem, it follows from (3.4) that

$$I(t) = \frac{S_0}{2} \left(\int_{|\omega|=1} d\omega \right) (\alpha + \beta)^{-\frac{n}{2}} t^{-\frac{n}{2}} + o(t^{-\frac{n}{2}}) \quad (3.5)$$

as $t \rightarrow +\infty$. Thus, one can get

$$I(t) \geq \frac{S_0}{4} \left(\int_{|\omega|=1} d\omega \right) (\alpha + \beta)^{-\frac{n}{2}} t^{-\frac{n}{2}}, \quad t \gg 1,$$

which implies the desired estimate from below.

The estimate from below of item (3) comes from the same idea used in [10, (2.4)]. In fact, if one sets the conical region $K \subset \mathbf{R}_\xi^n$ as

$$K := \left\{ \xi \in \mathbf{R}_\xi^n \mid \frac{\xi}{|\xi|} \cdot \frac{P_0}{|P_0|} \geq \frac{1}{2} \right\},$$

then one can observe that

$$\begin{aligned} & \int_{\mathbf{R}^n} \left| \frac{\xi}{|\xi|} \cdot P_0 \right|^2 e^{-t(\alpha+\beta)|\xi|^2} |\cos(\gamma t|\xi|)|^2 d\xi \geq \frac{|P_0|^2}{4} \int_K e^{-t(\alpha+\beta)|\xi|^2} |\cos(\gamma t|\xi|)|^2 d\xi \\ &= \frac{|P_0|^2}{4} \int_0^\infty \int_{K \cap \{|\xi|=r\}} e^{-t(\alpha+\beta)r^2} |\cos(\gamma tr)|^2 dS dr \end{aligned}$$

$$\begin{aligned}
&= \frac{c(n)|P_0|^2}{4} \int_0^\infty r^{n-1} e^{-t(\alpha+\beta)r^2} |\cos(\gamma tr)|^2 dr \\
&= \frac{c(n)|P_0|^2}{8} \left(\int_0^\infty r^{n-1} e^{-t(\alpha+\beta)r^2} dr + \int_0^\infty r^{n-1} e^{-t(\alpha+\beta)r^2} \cos(2\gamma tr) dr \right),
\end{aligned}$$

where the constant $c(n) > 0$ is the area of $K \cap \{|\xi| = 1\}$. So, one can proceed similar computations to item (2) in order to check the desired estimate from below of (3) by using the Riemann-Lebesgue Theorem. Finally, the estimate from below of item (1) is derived by the same argument as just above more easily, and for this we shall omit its detail. \square

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